Constructing Graphical Representations: Middle Schoolers' Intuitions and Developing Knowledge About Slope and Y-intercept

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Middle-school students are expected to understand key components of graphs, such as slope and y-intercept. However, constructing graphs is a skill that has received relatively little research attention. This study examined students' construction of graphs of linear functions, focusing specifically on the relative difficulties of graphing slope and y-intercept. Sixth-graders' responses prior to formal instruction in graphing reveal their intuitions about slope and y-intercept, and seventh- and eighth-graders' performance indicates how instruction shapes understanding. Students' performance in graphing slope and y-intercept from verbally presented linear functions was assessed both for graphs with quantitative features and graphs with qualitative features. Students had more difficulty graphing y-intercept than slope, particularly in graphs with qualitative features. Errors also differed between contexts. The findings suggest that it would be valuable for additional instructional time to be devoted to y-intercept and to qualitative contexts.

Graphical representations of functions are integral to algebra and important in students' mathematics education. Yet, many middle- and high-school students have a limited understanding of graphs (Blume & Heckman, 2000; Swafford & Brown, 1989). Studies have shown that conveying information with graphs and extracting information from graphs are often difficult for students (e.g., Donnelly & Welford, 1989; Eraslan, 2008; Padilla, McKenzie, & Shaw, 1986; Swatton & Taylor, 1994; Wainer, 1992) and that these difficulties span many levels of education (Wainer, 1992; Wavering, 1989).

Many studies assessing students' understanding of graphing have focused on students' abilities to interpret graphs (e.g., Friel, Curcio, & Bright, 2001; McKenzie & Padilla, 1986; Swatton & Taylor, 1994; Wainer, 1992) or to translate among various representations including graphical representations (Brenner et al., 1997; Moschkovich, Schoenfeld, & Arcavi, 1993). An understanding of graphing, however, involves not only interpretation and translation but also construction. Although technological advances such as graphing calculators and computer programs promote exploration and understanding of functions (e.g., Ainley, Nardi, & Pratt, 2000; Botzer & Yerushalmy, 2008; Hennessy, Fung, & Scanlon, 2001; Kieran, 2001; Levert, 2003; Nicolaou, Nicolaidou, & Zacharia, 2007; Noble, Nemirovsky, Dimattia, & Wright, 2004; Schwartz & Hershkowitz, 1999), they do not always help students gain a full understanding of function (Schoenfeld, Smith, & Arcavi, 1993).

Students' ability to construct graphs by hand (e.g., to create a graph from a table of values) has received relatively little research attention (Leinhardt, Zaslavsky, & Stein, 1990) with just a handful of studies focusing solely on graph construction (Krabbendam, 1982; Kramarski & Mevarech, 1997; Mevarech & Kramarski, 1997; Wavering, 1985). This research indicates that students may find graph construction difficult because they hold misconceptions about graphs, such as confusing height and slope, reading and constructing graphs point-wise (each point individually), and considering the graph as an icon (a literal picture of that situation) (see Leinhardt et al., 1990 for a review). These deficiencies may arise with a lack of knowledge of concepts, negative transfer from one representation to another, or confusion between the process of graphing and the graph as a product.

Graph construction also receives relatively little instructional attention in math curricula (Demana, Schoen, & Waits, 1993). The present study focuses on students' abilities to construct graphs.

According to the National Council of Teachers of Mathematics (2000), by the end of middle school, students are expected to understand key components of graphs, such as slope and y-intercept. Although some research has sought to reveal what students understand about these concepts (e.g., Lobato, Ellis, & Muñoz, 2003; Moschkovich, 1998; Turner, Wilhelm, & Confrey, 2000), there exists very little data that compares students' understanding of slope and y-intercept, and how that understanding changes across grade levels.

From one perspective, understanding slope seems to make higher demands on students' cognitive capacity than y-intercept. Understanding and graphing slope requires that a student be able to track two variables and the nature of their covarying relationship. Understanding and graphing y-intercept, on the other hand, involves attending to the value of one variable when the other variable is zero, essentially one variable at a single point. Furthermore, research has shown that students tend to read graphs by focusing on the individual points marked (Kerslake, 1981; Moschkovich et al., 1993; Yerushalmy & Schwartz, 1993). To determine slope, a student must work with a minimum of two points, whereas to determine y-intercept, only one point is required. For these reasons, slope may be harder to grasp and therefore more difficult to graph accurately than y-intercept.

Another perspective suggests, however, that slope may be easier to grasp than y-intercept. Students may have an intuitive understanding of slope from their real-world experiences with covariation, such as in cooking (for every egg, 1 cup of flour) or shopping (for every additional apple, it costs 50¢ more), and they may connect this prior experience with their classroom lessons in slope (e.g., Schlieman, Carraher, & Ceci, 1997). Students may have less experience with y-intercept outside of the classroom and therefore have fewer intuitions about the concept. Students may also have difficulty connecting real-world and classroom experiences with the concept of y-intercept (Davis, 2007). In some situations, it may be hard for students to find an "initial value." For example, students in one study found it difficult to plot a y-intercept when graphing the linear relationship between the number of scoops of ice cream in an ice cream cone and the amount of money it costs (Davis, 2007). Students were reluctant to plot a point where the ice cream cone would be purchased without ice cream (the y-intercept), believing that no one would ever purchase the cone alone in the real world.

Thus, informal learning through everyday life suggests that students may have stronger intuitions about slope and find it easier than y-intercept to represent in a graph. The present study will investigate students' performance graphing both of these concepts (slope and y-intercept).

Not only is it theoretically important to know which of these concepts is more difficult for students, but it is also important for practical reasons. Knowing how difficult it is for students to represent slope and y-intercept in a graph can bring to attention the need to understand *why* these concepts may be difficult for students to learn. With this knowledge, instruction can be adapted to best serve students' needs.

Regardless of whether slope or y-intercept is more difficult, by the end of eighth grade, students are expected to be able to interpret and construct graphs of linear functions. Instruction in linear functions now often begins in seventh grade (e.g., in *Connected Mathematics Program* [*CMP*] [Lappan, Fey, Fitzgerald, Friel, & Philips, 1998], the curriculum used in the school participating in this study). Prior to formal instruction in linear functions, students typically have experience with tables, bar graphs, and frequency plots to organize and represent data. With this prior experience and real-world experience, students in sixth grade may have intuitions about graphing slope (e.g., increase in cost with each additional apple, steepness of a hill) and y-intercept (e.g., height of a student at the beginning of the school year). The present study focuses on sixth-, seventh-, and eighth-grade students, which allows us to examine students' performance graphing linear functions both before and after formal instruction. Sixth-grade students' performance may reveal insights into their intuitions about graphing y-intercept and slope, while seventh- and eighth-grade students' performance may indicate how instruction has shaped their ability.

Another goal of this research is to explore students' understanding of slope and y-intercept in graphs with two different types of features: quantitative and qualitative. Graphs with quantitative features (see Figure 1 for an example) have specific values or increments marked for each variable (e.g., week # 1, 2, 3, etc.). Graphs with qualitative features (see Figure 2 for an example), though they may have axes labeled (e.g., weeks, money earned, etc.), do not display any specific values for any variable. In a qualitative graph, no numbers or values are attached to any variable, so there is no numerical information to use when constructing or interpreting the graph. Instead, students must look at the general trend of the graph and observe the pattern of covariation between two variables (Leinhardt et al., 1990). Just as research has focused little on students' abilities to construct graphs, qualitative interpretations of graphs are also underrepresented in math curricula and often experienced only in science units (Leinhardt et al., 1990).

When students examine graphs, they tend to concentrate on one point or several points rather than the more global structure of the graph (Bell & Janvier, 1981; Wainer, 1992). Consequently, students may interpret graphs as

Jamie is saving money. She has saved $7 so far and plans to save $3 each week. Draw a graph that shows the amount of money Jamie will have after each week.

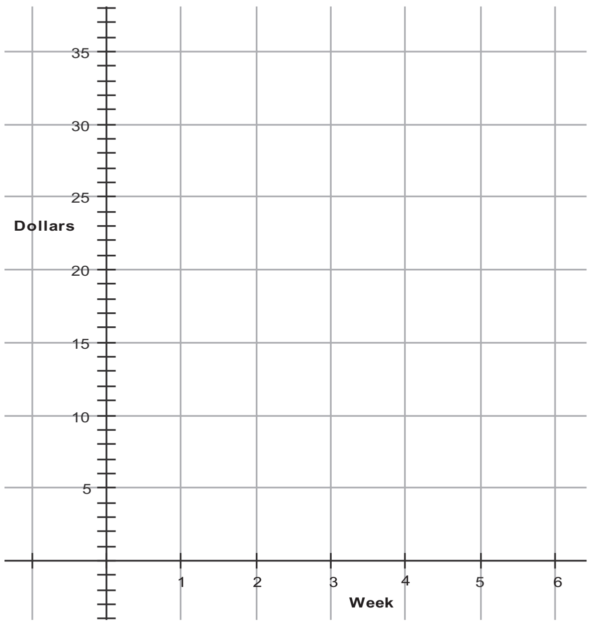
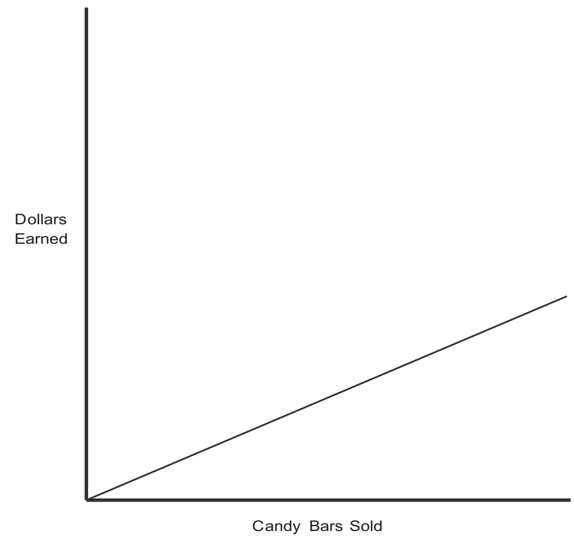


Figure 1. Quantitative item.

The school band is selling candy bars to raise money for their trip to the Dells. The following graph shows the relationship between the number of candy bars sold and the amount of money the band has earned.



a) If the band earned twice as much money for each candy bar, the graph would look different. Show what this would look like on the graph above and label is “A”.

b) If the band received a $50 gift from a parent when they began selling candy bars, the graph would also look different. Show what this would look like on the graph above and label it “B”.

Figure 2. Qualitative item.

they would on tables, focusing on specific pieces of information with little attention to the “bigger picture” of the underlying situation (Dugdale, 1993; Leinhardt et al., 1990). Oftentimes, when students are presented with graphs, they are asked to plot, read, or consider specific points on the graph. This “pointwise” approach is distinct from a more global approach in which students might be asked to attend to the general shape of the graph (e.g., noticing that the function creates a line). As students are typically presented with graphs with quantitative features and encouraged to calculate specific quantities for slope or y-intercept, they may be missing opportunities to recruit a global approach to making sense of graphs.

Graphs presented with only qualitative features (i.e., no numerical values) might help students draw upon their common sense and reality-checking strategies (Goldenberg, 1987; Krabbendam, 1982). Students’ intuitions about physical phenomena, however, may incorrectly influence their reading of qualitative graphs, promoting “iconic” interpretations (Nemirovsky & Rubin, 1991; Noble & Nemirovsky, 1995; Stylianou, Smith, & Kaput, 2005). For example, when discussing the steepest point of a graph, students may liken the curve to their knowledge of hills and select the top of the hill as the steepest point instead of determining where the curve has the greatest slope. As little empirical research exists comparing students’ performance constructing quantitative and qualitative graphs, one aim of this research was to examine whether quantitative and qualitative features influence students’ ability to construct graphs.

In brief, this study addresses two main questions: (1) How successful are students at graphing slope and y-intercept, and how does this performance change with grade level? and (2) Do qualitative or quantitative features of a graph influence students’ ability to graph slope and y-intercept?

Method

Participants were 180 middle-school students (59 sixth graders, 65 seventh graders, and 56 eighth graders) from a small urban district in the Midwest. The school’s student population was 38% minority and 41% low income.

Connected Mathematics was the curriculum used for instruction in all three grades. Students completed the assessment in the spring (mid-March), by which time sixth-grade students had received instruction (through *CMP*’s *Data About Us*) on how to represent data using line plots, bar graphs, stem-and-leaf plots, and coordinate graphs. Students in seventh grade had spent classroom time interpreting and constructing tables and graphs (by hand and with graphing calculators) to describe linear relations between variables (*Variables and Patterns*). These students had also received instruction on the relation between tabular and graphical linear patterns through discussion of slope, and y-intercept (*Moving Straight Ahead*). By the spring of eighth grade, students had received lessons on how to recognize and describe linear and nonlinear patterns in tables, graphs, and symbolic equations (through five of *CMP*'s instructional books designed for eighth grade).

The data that are the focus of this paper consist of students' responses to a subset of items on a written assessment that targeted their knowledge of functions and their fluency with different mathematical representations. This study addresses students' responses to the two graph construction items. Both of the items assessed students' ability to construct a graphical representation of a function that was described verbally (refer to Figure 1 for the *quantitative* item and Figure 2 for the *qualitative* item).

Students' graphical constructions for each item were scored in terms of their display of slope and y-intercept (referred to as "graphing slope and y-intercept" in this paper). For slope, on the *quantitative* item for which a grid was provided, students' graphs were expected to display the correct relationship between x and y (i.e., with each increasing week, an increase of $3). Straight lines, linear points, or bars were all accepted as correct (as the story could be interpreted as a continuous or discrete situation) if they depicted the correct relationship. For part A of the *qualitative* item (based on a problem in *CMP*'s *Moving Straight Ahead*, p. 44), any smooth linear graph drawn above the original line was categorized as a correct response. Incorrect responses for both items included all other linear graphs depicting any other slope and all nonlinear graphs.

With respect to y-intercept, for the *quantitative* item in which increments were marked, students' graphs were expected to account for the $7 received before the first week. Correct responses included graphs that marked the y-intercept (0,7) or began the line at a correct point (e.g., 1,10) from which the correct y-intercept could be inferred. For part B of the *qualitative* item, any linear graph starting from above the origin on the y-axis was coded as depicting correct y-intercept understanding. Incorrect responses included graphs that started at any other specific y-intercept (including incorrect y-intercepts as a result of changing the x-intercept) or at the origin (0,0).

Reliability of coding was assessed by having a second coder rescore 20% of the data. Agreement for scoring students' responses for slope accuracy was 100% for the *quantitative* item and 90% for the *qualitative* item. Agreement for scoring y-intercept accuracy was 100% for the *quantitative* item and 97% for the *qualitative* item. Discrepancies between coders were resolved through consensus meetings, and the resulting consensus codes were used for data analysis.

The design of this study therefore includes three variables: one between-subject factor (grade: sixth, seventh, or eighth) and two within-subject factors (graph type: qualitative or quantitative; concept: slope or intercept). Our aim was to understand how these three variables might affect students' performance in graphing linear equations.

Results

We analyzed the data by modeling the repeated measures logistic data with weighted least squares regression (Stokes, Davis, & Koch, 2000). We calculated a score for each student for each of our repeated trials (slope in a quantitative graph, slope in a qualitative graph, intercept in a quantitative graph, and intercept in a qualitative graph). Students received a 1 for accurate responses and a 0 for inaccurate responses.

How Successful Are Students at Graphing Slope and Intercept, and How Does This Performance Change With Grade Level?

As seen in Figure 3, there was a significant interaction between grade level and concept, $Q_W = 24.71$, $df = 2$, $p < .0001$. Students were less successful at graphing y-intercept compared with slope, particularly in sixth and seventh grades. The difference between performance on slope and y-intercept decreased by eighth grade. As seen in the figure, sixth-grade students appear to have good

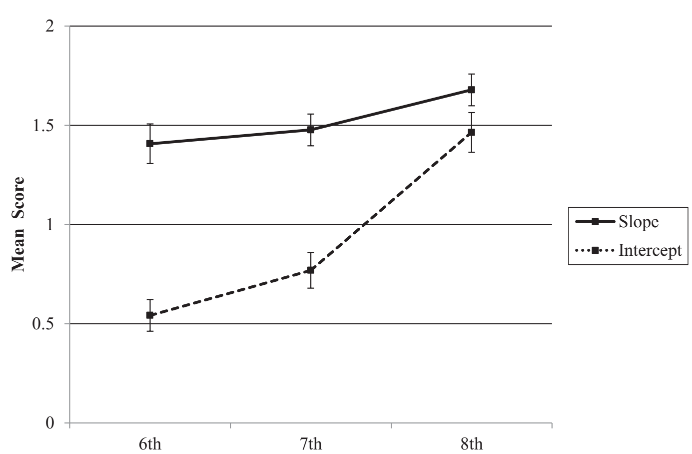


Figure 3. Mean scores for graphing slope (solid line) and y-intercept (dotted line) across two items as a function of grade.

Table 1
Proportion of Correct Responses by Grade, Concept, and Graph Type

Concept	Graph Type	Sixth	Seventh	Eighth
Slope	Quantitative	0.71	0.67	0.82
	Qualitative	0.69	0.80	0.85
Intercept	Quantitative	0.42	0.42	0.71
	Qualitative	0.12	0.36	0.75

intuitions about slope, but not about y-intercept, despite having had little or no formal instruction about graphs of linear functions. An impressive 70% of the sixth graders were able to correctly represent slope graphically. For y-intercept, however, students' responses revealed uncertainty about how to represent it graphically. By eighth grade, performance in graphing both slope and y-intercept improved, although the graphical representation of y-intercept still lagged behind that of slope. Even though both concepts are covered in seventh- or eighth-grade *CMP* units (e.g., *Moving Straight Ahead*, *Thinking with Mathematical Models*, *Say It with Symbols*), by mid-March (when this assessment was given), nearly a quarter of eighth-grade students were unable to correctly graph slope, and a third were unable to correctly graph y-intercept.

The main effect of concept was significant, with students more successful at graphing slope than y-intercept, $Q_W = 119.27$, $df = 1$, $p < .0001$. Not surprisingly, students' performance also increased across grade levels, $Q_W = 35.27$, $df = 2$, $p < .0001$(also reliable when excluding students who did not respond), regardless of the concept they were graphing.

Do Qualitative or Quantitative Features of a Graph Influence Students' Ability to Graph Slope and Y-Intercept?

Next, we compared students' responses in graphing slope and y-intercept between the quantitative and qualitative items (see Table 1 for means and standard deviations by grade, concept, and graph type). The interaction between graph type and concept was significant, $Q_W = 9.41$, $df = 1$, $p = .002$. When graphing slope, the type of graph did not affect performance (see Figure 4). When graphing y-intercept, however, as seen in Figure 5, sixth-grade students had more difficulty when the graph was presented with qualitative features than with quantitative features. We consider possible explanations for this finding in the discussion.

In order to understand better how student performance in graphing slope and y-intercept differs between quantitative and qualitative graphs, we next examined students' errors.

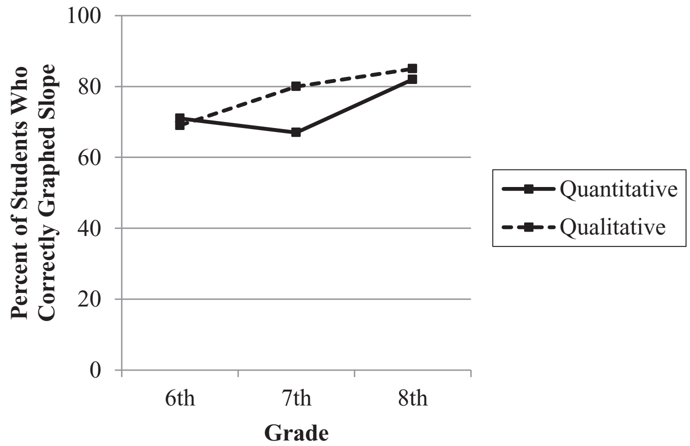


Figure 4. Percent of students who correctly graphed slope across the two types of graphs (quantitative and qualitative).

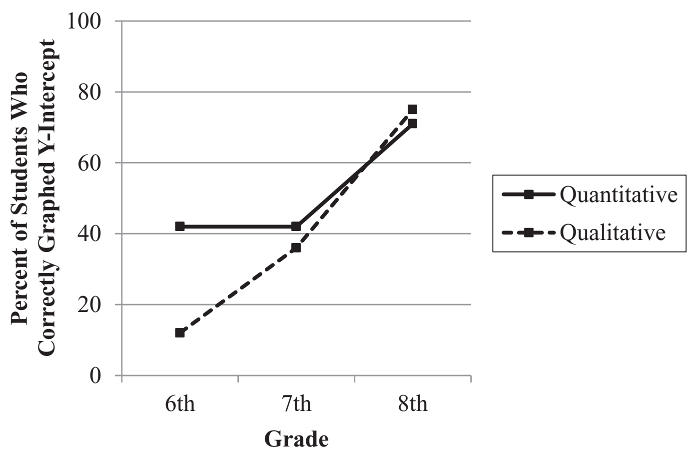


Figure 5. Percent of students who correctly graphed y-intercept across the two types of graphs (quantitative and qualitative).

What Types of Errors Did Students Make?

Slope errors. Closer examination of students' responses in graphing slope and y-intercept revealed that the two types of graphs (quantitative or qualitative) afforded different types of errors (see Table 2 for a description of errors and Table 3 for percent of students making errors of each type by grade). With the quantitative graph, about 10% of students (in the sample as a whole) failed to respond. When students did respond, their errors in graphing slope most often involved arithmetic errors at some point in the calculation process. Because students often correctly calculated several points before making an arithmetic error, they sometimes produced nonlinear graphs (see Figure 6a).

In the graph with qualitative features, no-response errors were particularly common (18.3% of students overall). When students did respond, their errors in graphing slope most often involved constructing a line with an

Table 2
Description of Errors by Concept and Graph Context

Concept	Context	Error Type	Description
Slope	Quantitative	Arithmetic error	Error in calculating one point from another that led to a nonlinear graph, e.g., with a slope of 2, points include (2,2), (3,4), and (4,7)
		Slope magnitude error	Produced a line with an incorrect slope
	Qualitative	Under	Slope decreased from original line
		Same	Slope not changed from original line
Y-intercept	Quantitative	Time error	Plotted the initial amount of money at week 1
		Y-intercept magnitude error	Produced graphs where an incorrect y-intercept was drawn or could be inferred
	Qualitative	Origin	Graphed a line passing through the origin
		Horizontal Axis	Produced a line with a y-intercept to the right of the original line

Table 3
Percent of Students Making Errors of Each Type for Quantitative and Qualitative Graphs by Grade

Concept	Context	Error Type	Sixth N = 59	Seventh N = 65	Eighth N = 56	Overall N = 180
Slope	Quantitative	Arithmetic error	3.4%	10.8%	8.9%	7.8%
		Slope magnitude error	1.7%	9.2%	1.8%	4.4%
		No response	16.9%	9.2%	5.4%	10.6%
		Other	6.8%	3.1%	1.8%	3.9%
		Total	28.8%	32.3%	17.9%	26.7%
	Qualitative	Under	0%	3.1%	0%	1.1%
		Same	1.7%	1.5%	3.6%	2.2%
		No response	28.8%	15.4%	10.7%	18.3%
		Total	30.5%	20.0%	14.3%	21.7%
Y-intercept	Quantitative	Time error	8.5%	23.1%	10.7%	14.4%
		Y-intercept magnitude error	25.4%	23.1%	10.7%	20%
		No response	16.9%	9.2%	5.4%	10.6%
		Other	6.8%	3.1%	1.8%	3.9%
		Total	57.6%	58.5%	28.6%	48.9%
	Qualitative	Origin	47.5%	38.5%	14.3%	33.9%
		Horizontal axis	0%	4.6%	0%	1.7%
		No response	40.7%	21.5%	10.7%	24.4%
		Total	88.1%	64.6%	25.0%	60.0%

incorrect slope magnitude—either decreasing the slope from the original line (see Figure 6b) or not changing the slope at all.

Y-intercept errors. Errors in graphing y-intercept also differed between the qualitative and quantitative graphs. When asked to construct a graph with quantitative features, as noted earlier, about 10% of students (in the sample as a whole) failed to respond. When students did respond, one common error involved is students constructing a graph with a y-intercept falling somewhere along the y-axis but at the incorrect value of y. Another common error that we identified in this dataset involved plotting the initial amount of money (the $7 Jamie has to start with) not on the y-axis but one increment over, corresponding with the first week (see Figure 7a). This resulted in a corresponding time error. Students were disinclined to plot the initial $7 as a value along the y-axis. A few students drew an additional line segment connecting the point (1,7) to the origin, although this created a nonlinear curve.

For the qualitative item, no-response errors were common (24.4% of students overall), and confusion with the origin led to another frequent type of y-intercept error. Consistent with the literature (Leinhardt et al., 1990), students often drew a line from the same point as the line provided on the graph (the origin) but changed the slope of the line (see Figure 7b). Again, this error reveals a tendency for students to begin a line at the origin. Students may begin the line at the origin for a number of reasons: out of a "default" tendency because they do not understand how to represent a change in y-intercept on the graph or

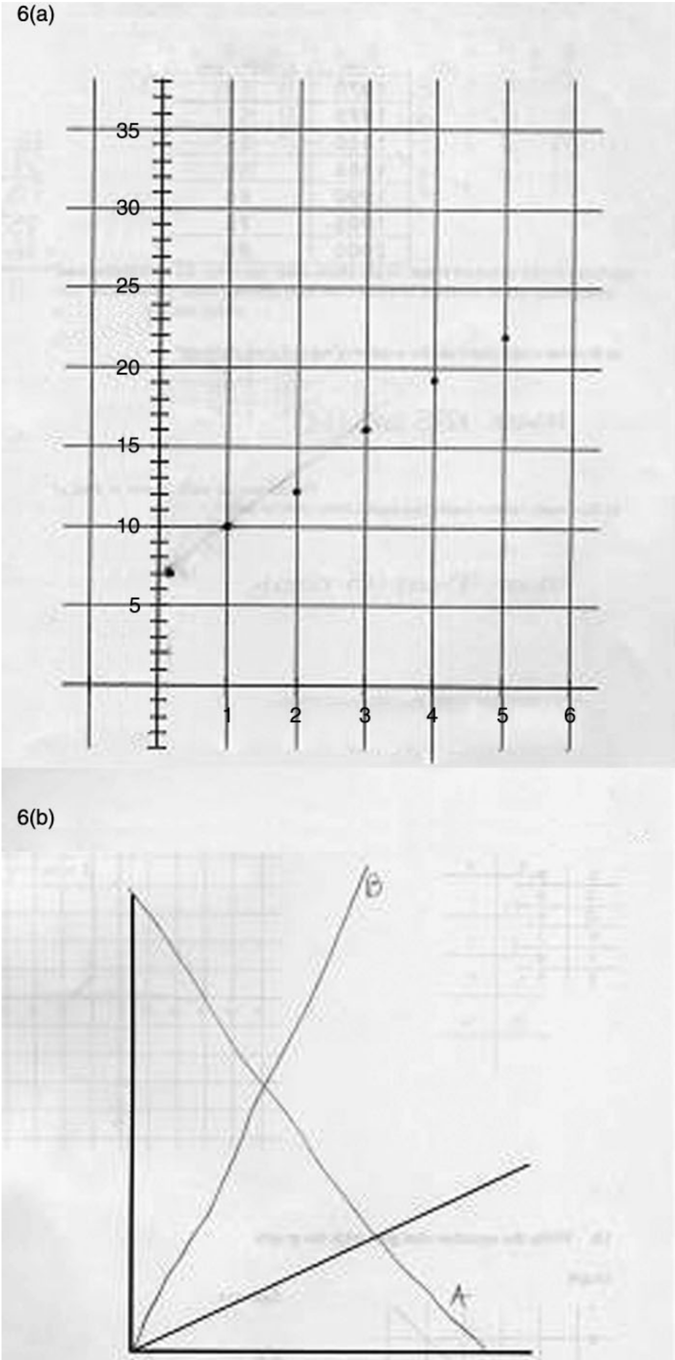


Figure 6. Examples of student's work revealing slope errors with quantitative and qualitative graphs. *(a)* Quantitative graph: arithmetic error (top); *(b)* Qualitative graph: decreasing slope error (bottom).

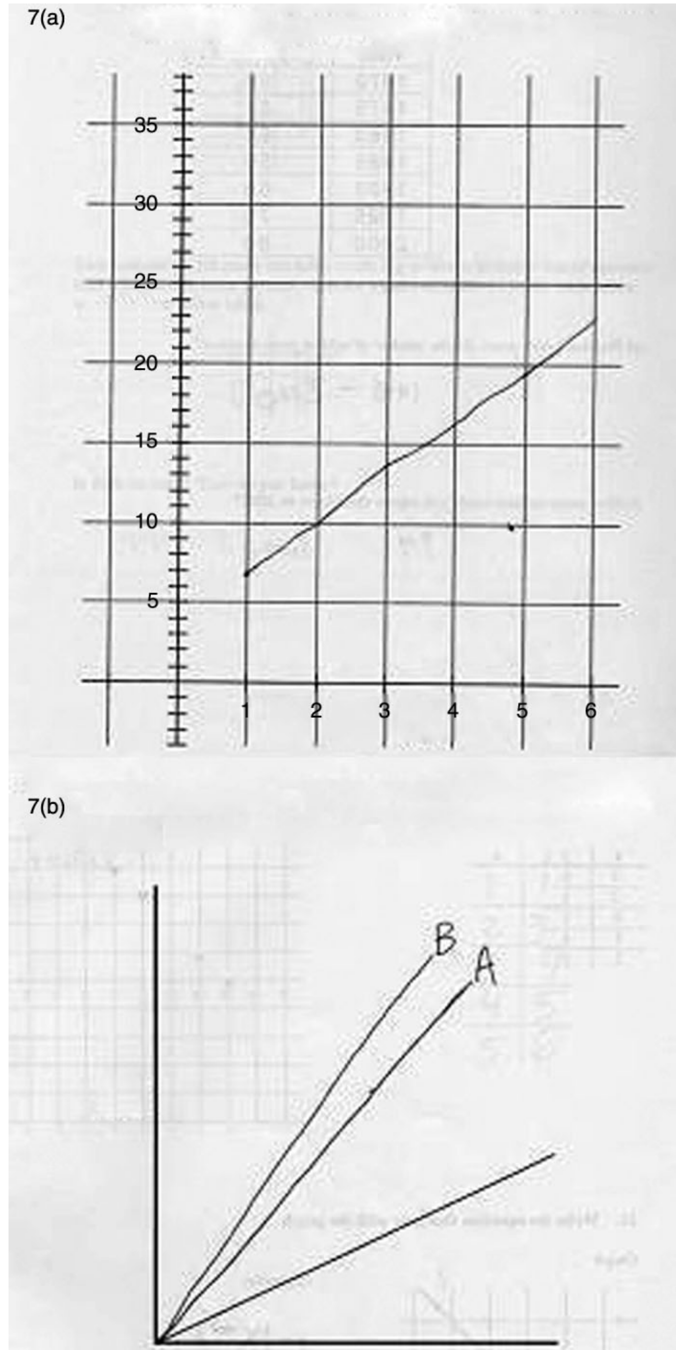


Figure 7. Examples of student work revealing y-intercept errors with quantitative and qualitative graphs. *(a)* Quantitative graph: plotting one time increment over (top); *(b)* Qualitative graph: plotting from origin and changing slope (bottom).

because they do not understand the difference between variables being linearly related and proportionally related. Regardless of the reason, however, students did recognize that with an additional $50, the line should look different. Students then changed the graph in a way with which they were familiar (slope), and hence, the resulting error involved both y-intercept and slope confusion. More broadly, students tend to use the origin as the y-intercept, regardless of the situation being represented.

Discussion

Understanding of Slope and Y-Intercept

This study focused on one aspect of students' ability—graph construction. We recognize that understanding of slope and y-intercept would best be assessed in multiple ways; however, we believe that performance on a task such as graph construction can provide some insight into students' understanding of slope and y-intercept. The graph construction tasks included in this study require some understanding of slope and y-intercept, and so successful performance on these tasks reflects some (potentially incomplete) understanding of these concepts. We discuss the results of this study from this perspective.

Overall, students had more difficulty graphing y-intercept than slope. In sixth grade, before students receive formal instruction in graphing linear functions, students' responses reflected accurate intuitions about slope more frequently than y-intercept. Although performance in slope and y-intercept both improved across grades, even in eighth-grade students' graphing of y-intercept lagged behind slope.

Students' ability to construct graphs with correct slope may be the result of their successfully transferring prior knowledge of slope in tables to understanding slope in graphs. Students may also relate their real-world experiences with covariation, or with hills or mountains to their understanding of graphing slope. Given recent research on students' understanding of covariation (Blanton & Kaput, 2004), as well as efforts to integrate algebraic ideas into elementary school mathematics curricula (Kaput, 1998; Olive, Izsak, & Blanton, 2002), it is perhaps unsurprising that students have a fairly good ability to graph slope even before the topic is addressed in formal instruction.

The results indicate, however, that middle-school students have a poor understanding of graphing y-intercept. Students entering sixth grade may have little understanding of y-intercept as a concept or may find it difficult to transfer their knowledge of y-intercept from other representations (e.g., tables) to graphs.

It is also possible that students are unable to relate y-intercept on a graph to their real-world understanding of y-intercept (Davis, 2007), such as the height of a student at the beginning of the year. Intercept errors may also reflect students' real-world knowledge and experience of independent variables such as time, where the initial value is almost always zero. Students' early experiences with graphs (e.g., distance vs. time) include many functions where both the x- and y-intercepts are equal to zero (Kalchman & Koedinger, 2005), which may also contribute to this error.

Another possibility is that weakness in graphing y-intercept may stem from an unequal balance in instructional attention between the concepts of slope and y-intercept. Often when graphing is introduced in the curriculum, although both slope and y-intercept are covered, much more instructional time is spent on the concept of slope (e.g., *CMP*, *Grade 7*, *Variables and Patterns*). In the *CMP* curriculum, at least, the concept of y-intercept receives less instructional time in the classroom. This is then reflected in students' stronger abilities to graph slope than y-intercept.

Constructing Qualitative vs. Quantitative Graphs

Students, particularly sixth-grade students, had more difficulties with y-intercept when constructing qualitative graphs. Overall, sixth-grade students displayed poor performance in graphing y-intercept, but they were somewhat successful with the more familiar quantitative graph. With the help of quantitative features of the graph, students were able to focus on local aspects of the graph and pick out specific points and values. Past research suggests that there may be an advantage for processing information within a local region for children at this age (Moses et al., 2002), although with repeated exposure to such quantitative graphs that allow a local focus of attention, concepts like slope and y-intercept often remain limited to specific values.

The differences in the types of slope errors produced between quantitative and qualitative graphs may reflect the focus of attention required for each type of graph. In a graph with quantitative features, when using an additive strategy in calculating slope, for example, a student might count $3 (vertically) for every 1 week (horizontally). With a great deal of focus on calculating the specific points and increments, the student may never come to view the graph as a whole. Instead, the student may view graphing as a process of calculating point values such that for every value of x, there is a corresponding y value (Moschkovich et al., 1993). This perspective may account for the nonlinear curves students sometimes produced for the quantitative graphs as a result of arithmetic errors. Focused on the process of calculating a collection of points to make a line, students may neglect to see the graph as a whole. In fact, past research has noted that many students do not ever view a function or graph as a whole entity or object (Yerushalmy & Schwartz, 1993). In some cases, the focus of learning does not go beyond producing graphs through some process or action. Students' focus on the graph as a collection of local points (and not a line as a whole) may be due in part to individual differences in metacognitive skills such as cognitive control (i.e., processes responsible for planning, initiating appropriate actions, and inhibiting inappropriate actions). Students with weaker metacognitive skills may rarely take the extra time, after plotting individual points, to check their graphs by viewing the line as a whole, and therefore, they may not recognize that they have produced a nonlinear graph.

If, on the other hand, students are not provided with quantitative features such as increments marked on the graph, the focus of attention is shifted away from individual points, and students are more likely to view the graph globally or as a whole. Indeed, nonlinear curves were seldom seen in students' responses to the qualitative item in this study. Students constructing graphs with qualitative features did make slope errors, but these errors typically maintained an intact line with an altered slope. With a qualitative graph, students' errors revealed that they viewed the graph or line as a whole entity, one that could be "picked up and moved."

Because of the focus of instruction on local aspects of graphing rather than a more global view (Bell & Janvier, 1981; Demana et al., 1993), a process perspective on graphing is often sufficient for success in the classroom. Research on graphing, however, makes it clear that both object and process perspectives on graphing are essential in learning about functions and graphs (Swafford & Brown, 1989). Moschkovich et al. (1993) argued that "developing competency with linear relations means learning which perspectives and representations can be profitably employed in which context, and being able to select and move fluently among them to achieve one's desired ends" (p. 72).

Effects of Grade Level on Graph Construction

When graphing y-intercept with quantitative and qualitative graphs, student performance improved considerably from seventh to eighth grade. For an explanation of this jump in performance, we can look to the students' curriculum. In *CMP*, students in seventh grade learn about slope and intercept, and the relationship between tables and graphs through two books in their curriculum set. In eighth grade, they learn to recognize not only linear but also nonlinear functions in graphs, and they learn about the relationships among tables, graphs, and equations. Eighth-grade students learn these concepts through five books in their curriculum set. Students in eighth grade receive more instruction on functions and graphing than students in seventh grade, and this is reflected in the students' performance on both quantitative and qualitative graph construction.

There is also an increase in performance between sixth and seventh grade in graphing intercept on the qualitative graph (see Figure 5). Students in sixth grade receive very little instruction on coordinate graphs and even less instruction on graphs with qualitative features. With little knowledge, they are hesitant to construct graphs, as illustrated by the large proportion of students in sixth grade (24 out of 59 or 41%) who did not respond when asked to graph the y-intercept on a qualitative graph. By seventh grade, however, students have more experience with graphing linear functions, and they are more likely to attempt a response (only 22% of seventh graders did not respond) when asked to construct a graph with qualitative features. The increase in students responding to these items from sixth to seventh grade mirrors the increase in performance in correctly graphing y-intercept on the qualitative graph.

Practice graphing linear functions with quantitative as well as qualitative graphs could provide students opportunities to exercise both process and object perspectives, and to switch from one perspective to another between and even within graphs. With this flexibility, students would be less likely to make errors such as producing nonlinear graphs for a linear function.

Implications for Instruction and Directions for Future Research

These data indicate that students have difficulty in graphing y-intercept, and they are less successful in correctly graphing slope and intercept in qualitative graphs. These findings suggest that it would be valuable for additional instructional time to be devoted to y-intercept and to qualitative contexts. A greater emphasis on qualitative graphs might also be valuable in encouraging a more global approach to graph construction and interpretation.

Future research should address several remaining questions. It would be interesting to conduct interviews with students solving these problems to find out why students find it difficult to graph y-intercept. Questions such as "How much money does the band have at week 0?" and "Where is the $50 gift from the parents accounted for in your graph?" could help to determine where students struggle in their understanding and representations of y-intercept in graphs.

In addition, it would be interesting to examine how ninth and tenth graders fare in solving these problems. High-school students may have a better understanding of the concepts of slope and y-intercept in graphs, though it is possible that performance could remain stable on these tasks after eighth grade. Although these concepts should be understood by the end of eighth grade, it would be interesting to know whether students in high school still struggle with them.

Finally, an analysis of textbooks could reveal to what extent graphing slope and intercept are discussed in math and science classrooms. This study focused on one curriculum (*CMP*), but it would be interesting to see to what extent other types of curricula devote time specifically to slope and y-intercept.

With regard to teaching of graph construction, we also have a specific suggestion. In order to emphasize the concept of y-intercept, linear functions could be taught using the equation $y = b + mx$ instead of the traditional $y = mx + b$. With the y-intercept represented in the beginning of the right side of the equation, students might attend to this concept first, and they may give more consideration to understanding what this term means in constructing graphs.

Conclusion

In summary, the present study showed that middle-school students have more difficulty graphing y-intercept than slope. Although sixth-grade students' graph constructions revealed accurate intuitions about slope, students' performance in graphing y-intercept lagged behind graphing slope even in eighth grade. Errors in students' responses further revealed that students had more difficulty graphing y-intercept when the graph displayed qualitative features as opposed to quantitative features. Taken together, these results indicate that although both slope and y-intercept are fundamental concepts, students are not as successful at graphing y-intercept as they are at graphing slope. Instruction should provide students with opportunities to develop a better understanding of y-intercept and should also provide students opportunities to construct graphs with quantitative as well as qualitative features.

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