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The Developmental Role of Intuitive Principles in Choosing Mathematical Strategies

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This study investigated the relation between the development of understanding principles that govern a problem and the development of mathematical strategies used to solve it. College students and 2nd, 5th, 8th, and 11th graders predicted the resulting temperature when 2 containers of water were combined. Students first estimated answers to the problems and then solved the problems using math. The pattern of estimated answers provided a measure of the intuitive understanding of task principles. Developmental differences in intuitive understanding were related to the type of math strategy students used. Analysis of individual data patterns showed that understanding an intuitive principle was necessary but not sufficient to generate a math strategy consistent with that principle. Implications for the development of problem solving are discussed.

Current models of problem solving propose that a person's conceptual or intuitive understanding is an important factor in solving a problem with formal methods such as mathematics. Conceptual or intuitive understanding involves the qualitative representation of the relevant relations among variables in a task. We call this type of understanding *intuitive,* following Brunswik (1956) and Hammond (1982; Hammond, Hamm, Grassia, & Pearson, 1987). Greeno, Riley, and Gelman (1984) proposed that this type of understanding, in which general principles of the task domain are represented, constrains and justifies performance.

Most models of problem solving represent intuitive understanding with knowledge structures such as schemas, productions, or principles that contain information about the problem domain. For example, models of children's counting explain developmental differences in performance through differences in knowledge of principles (Briars & Siegler, 1984; Gelman & Gallistel, 1978; Greeno et al., 1984). Understanding the one-to-one correspondence principle allows individuals to reject a counting procedure that assigns two numbers to one object. Note that one-to-one correspondence is a principle about the domain of number, not a counting procedure itself.

The use of principles that govern the task has also been shown to be important in more sophisticated domains. Experts group physics problems according to physics principles, but novices group problems according to surface features (Chi, Glaser, & Rees, 1982; Hardiman, Dufresne, & Mestre, 1989). Further, novices who mention principles when categorizing problems perform better when solving subsequent problems (Hardiman et al., 1989). Models of solving word problems also make a distinction between understanding the domain and the formal procedures used to solve problems (Briars & Larkin, 1984; Cummins, Kintsch, Reusser, & Weimer, 1988; Kintsch & Greeno, 1985).

All the approaches discussed above have a representation of the task domain that is distinct from the representation of the formal procedures used to solve the problem. We call these representations of the problem domain intuitive understanding. We propose that intuitive understanding of the task is represented in terms of principles that specify the relations between variables in the task. Domain-independent formal methods for solving problems, such as mathematics, are represented separately and are not part of understanding how the task works. We also propose that development of the intuitive representation is an important factor in explaining developmental differences in how individuals solve problems with formal methods.

The present study explores the relationship between the development of intuitive understanding and the use of formal strategies. In order to examine this relationship it is necessary to measure intuitive understanding independently of formal strategies. Most research on problem solving has not directly measured intuitive understanding. Some researchers have assumed that intuitive understanding is at ceiling for a particular type of task (e.g., Cummins et al., 1988; Kintsch & Greeno, 1985). Other researchers have used groups that are presumed to differ in their intuitive understanding, for example, novices and experts (e.g., Chi et al., 1982; Larkin, McDermott, Simon, & Simon, 1980; Reed, 1987), or have inferred differences in intuitive knowledge from patterns of formal problem solving performance (e.g., Gelman & Gallistel, 1978; Greeno et al., 1984). Other researchers have measured intuitive understanding using participants' judgments about the goodness of another person's formal problem solving (e.g., Briars & Siegler, 1984; Gelman, Meck, & Merkin, 1986; Siegler & Crowley, 1994). We present a method for measuring intuitive understanding independently of any formal problem solving. Measuring intuitive understanding separately from formal problem solving, either

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the participant's own or that of another person, allows an examination of: (a) the development of intuitive understanding without the influence of developmental differences in understanding formal strategies, and (b) the relation between people's understanding of the problem domain and how they solve the problem using formal strategies.

Assessment of Intuitive Understanding

In order to assess intuitive understanding of a domain independently of formal problem solving, it is necessary that individuals perform in the domain without using formal strategies. In the present study, we have participants estimate the answers to problems. In this task, which we call the intuitive task, the values of the variables are given pictorially and with verbal descriptions, rather than numerically. Participants make their estimates by physically adjusting a marker on an ungraduated scale. Reed (1987) proposed that estimation provides a good index of what people understand about a task.

Using this method, we measured intuitive understanding of a temperature mixture task in which participants are shown two containers of water each at a particular temperature. They are asked to judge what the temperature of all the water together would be after the water was combined. The content of one container, the initial container, is always the same (i.e., temperature and amount did not vary). The content of the other container, the added water, varies in a factorial design. Each participant's pattern of judgments is scored for consistency with the principles that govern the domain. (The principles are described below.)

Four Principles of Temperature Mixture

Past research has identified four principles that can be used to characterize intuitive understanding of the temperature mixture task. The first goal of the present study is to measure intuitive understanding of temperature mixture in terms of these principles and to examine whether these principles are explicitly used by participants as they generate mathematical solution strategies. Several lines of evidence suggest that intuitive understanding of mixture tasks is accomplished by understanding these principles. First, Strauss and Stavy (1982) reported that some participants justified their responses in a temperature mixture task and a sweetness mixture task by mentioning some of the principles we identify below. Second, similar principles have been proposed by a number of researchers working with different mixture tasks (Ahl, Moore, & Dixon, 1992; Moore, Dixon, & Haines, 1991; Reed & Evans, 1987; Strauss & Stavy, 1982). Third, developmental differences in performing the temperature mixture task are consistent with the hypothesis that individuals acquire the principles (Ahl et al., 1992; Moore et al., 1991). Ahl et al. (1992) and Moore et al. (1991) showed that different judgment patterns in the estimation task could be explained by the development of understanding these principles of temperature mixture. The four principles are explained below.

Above–Below is the principle that, because the temperature of the initial container is held constant, the final temperature should always be in the direction of the added temperature from the initial temperature. For example, when 20° water is combined with the 40° initial container the resulting final temperature must be colder than 40°.

The Range principle is that the final temperature must fall between the initial and added temperatures. For example, when 50° water is combined with the 40° initial container, the resulting final temperature must be between 50° and 40°.

The Crossover principle is the interaction between temperature and quantity: The greater the quantity of the added water, the greater its effect, but the direction of the effect depends on whether the added temperature is above or below that of the water in the initial container. For example, three cups of 60° water will result in a warmer final temperature than one cup of 60° water when each is combined with the initial container.

The Equal–Temperatures–Equal (ETE) principle states that when the initial and added water are the same temperature, there is no temperature change. For example, when 40° water is combined with the 40° initial container, the resulting final temperature must be 40°.

It is worth noting that Above–Below, Range, and ETE are logically related to one another. Above–Below specifies the relation between the answer and the initial temperature (i.e., whether the answer should be larger or smaller than the initial temperature). Range specifies the relation between the answer and both the initial and the added temperatures. ETE can be considered a special case of Range for trials where the initial and added temperatures are equal. The reason for including all three principles is that, despite the logical relations among the three principles, participants appear to use all three principles to understand the task.

Assessment of Formal Strategies

In addition to having each participant perform the temperature mixture task in the intuitive condition, we also had them perform the task in a numerical condition. In the numerical condition, the temperatures and quantities were specified in terms of numbers as well as pictorially. Participants were asked to use math to solve the problems and to think aloud while doing so.

Relation Between Intuitive Understanding and Generating Math Strategies

We looked for two types of evidence regarding the relation between intuitive understanding and use of math strategies. First, would participants sometimes spontaneously mention the principles in thinking aloud while attempting to solve the task with math in the numerical condition? This would be evidence that they use the principles when trying to generate a math strategy. Furthermore, participants should only mention principles that they understand as shown by their scores on the principles from the intuitive task. However, participants may *fail* to mention principles for a number of reasons. For example, they may not understand the principle or they may understand it implicitly but not have the ability to verbalize it, or they may simply omit mention of it.

Second, to assess the developmental relation between intuitive understanding and selecting math strategies, we compared our measures of intuitive understanding derived from the intuitive condition to the particular mathematical strategies used in the numerical condition. If intuitive understanding affects the generation of mathematical strategies, then different intuitive understandings should be evident for participants using different strategies. Previous research has suggested a causal role for intuitive understanding in generating math strategies. Hardiman, Well, and Pollatsek (1984) showed that giving feedback on an estimation task improved overall math performance on structurally similar problems. Ahl et al. (1992) showed that performing the intuitive task before the numerical task increased accuracy. These findings suggest that intuitive understanding may influence the generation of mathematical strategies. In this study, we record verbal protocols to examine this question more precisely.

Past work has shown that understanding of the principles increases developmentally from second grade through college (Ahl et al., 1992; Moore et al., 1991). However, there is considerable variability within age group, especially in the younger age groups. Because we were interested in how the development of intuitive understanding is related to developmental differences in math strategies, we sampled students from five different age groups between second grade and college.

Because we were interested in whether a systematic relation between intuitive understanding and math strategies existed at all, we attempted to create optimal conditions for participants to use their intuitive understanding while generating math. Clearly, if intuitive understanding is used in generating math, a large number of factors may influence whether it is used in any particular problem situation. In the present study we attempted, in two different ways, to create optimal conditions for participants to use their intuitive understanding while generating math. First, we always presented the intuitive task before the numerical task. Performing the intuitive task first should cue participants' intuitive understanding, making it more readily available during the numerical task (Ahl et al., 1992). Second, all the participants received instructions in the numerical condition which encouraged them to compare the two tasks. Half of the participants received additional instructions that encouraged them to first estimate the answer in the numerical task before solving the problem with math. Pointing out the similarity between the tasks and presenting the intuitive task first should help participants use their intuitive understanding during the strategy selection process.

In summary, we hope that by having a detailed measure of intuitive understanding and an exact accounting of what math strategies participants use, we may be able to begin to explain how the development of intuitive understanding influences the development of formal problem solving.

Method

Participants

One hundred sixteen students from five grades participated: 11 second graders, 26 fifth graders, 20 eighth graders, 29 eleventh graders, and 30 college students. Consent was obtained from all participants. Parental consent was obtained for the four younger groups. The four younger groups participated as volunteers from local public and parochial schools. College students received extra credit points. The elementary and high-school age participants were from a predominantly European American, small-size city (population approximately 200,000) in the Midwest. The schools were located in middle-class neighborhoods. The college students were from a large state university that also had a predominantly European American, middle-class population.

Materials and Design

All participants completed a temperature mixture task in each of two conditions, intuitive and numerical. In both conditions, participants were asked to predict the final temperature of water in a container, given the initial quantity and temperature of the water and the quantity and temperature of water added to the initial container. Two schematic water containers, one for the initial water and one for the added water, were used as stimuli. The schematic water containers were felt board 12 × 16 in. (30.5 × 41 cm). Blue felt strips of three different sizes were used to represent the quantities of water. Each water container was paired with a schematic thermometer to represent the water temperature. The schematic thermometers were 16 in. (41 cm) high. A movable marker indicated the water temperature on each thermometer.

In the intuitive condition, quantity and temperature were represented pictorially and described verbally (e.g., large amount, cold water). No numbers or graduations were presented to the participants on either the thermometers or containers in the intuitive condition. The ends of the thermometer were labeled with a drawing of a fire and a snowman to represent temperature extremes. The experimenter had numbers and graduations available on the back of each thermometer. The experimenter manipulated the marker on the added container's thermometer to specify the added temperature. The participants responded by adjusting the thermometer of the initial container.

In the numerical condition, quantity and temperature were represented pictorially but described numerically (e.g., 1 cup, 60°). The thermometers and containers were identical to the ones in the intuitive condition except that they were marked with numbers in the numerical condition. In the numerical condition, participants were asked to try to use mathematics to find the answer.

The intuitive task was always completed before the numerical task. The intuitive task was a 3 (Added Quantity) × 5 (Added Temperature) factorial design, and the numerical task was a 3 (Added Quantity) × 3 (Added Temperature) factorial design. The smaller factorial design was used in the numerical condition because students regard these as difficult math problems. Asking them to perform more than nine problems seemed excessive. The added temperatures in the intuitive factorial design were analogous to 20°, 30°, 40°, 50°, and 60° and were verbally labeled "very cold," "cold," "cool," "warm," and "very warm," respectively. The added temperatures in the numerical factorial design were 20°, 40°, and 60°. In both designs the added quantities were 1, 2, and 3 cups, which were described as a "small amount," "medium amount," and "large amount" in the intuitive condition. The initial temperature in both conditions was always 2 cups of 40° (described as a medium amount of cool water in the intuitive condition). Trials within each task were presented in one of five random orders.

Procedure

Participants were tested individually in sessions lasting approximately 30 min. The experimenter told the participant that he or she would be doing two types of problems about the temperature of some water when other water was combined with it. The intuitive task was then described. The experimenter explained that the felt boards represented containers of water. The participant was told that each container had a thermometer that went with it which indicated the temperature of the water in the container. The experimenter explained the thermometer to the participant and gave examples of extreme temperatures (i.e., very warm and very cold). The experimenter did not proceed until satisfied that the participant understood the response scale. The experimenter explained that the container on the participant's right (the initial water) would always start out with a medium amount of cool water. The other container (the added water) would have different amounts and different temperatures each time. Participants were asked to judge what the temperature of all the water together would be when the added water was combined with the initial water. Participants responded by adjusting the thermometer of the initial container to show the combined water temperature.

After completing the intuitive task, participants were read the instructions (either elaboration or no elaboration, as explained below) for the numerical task. The experimenter explained that the task would be almost identical to the one just completed but that this time the quantity and temperature would be given in numbers and that they should try to use math to find the answer. The experimenter explained the task again in the same way as in the intuitive condition except that numbers were now used. The experimenter familiarized the participants with the numbered thermometers and the numbers for the quantities of water. Paper and pencil were provided in the numerical condition to facilitate computation. All participants were asked to think aloud as much as possible during the numerical task. Participants were instructed to reason aloud and say what numbers they were using and what they were doing with them. Their verbalizations were tape recorded.

Participants performed the numerical task in one of two instruction conditions: elaboration or no elaboration. In the elaboration condition participants were told to estimate the answer before computing with math by translating the numerical values into intuitive values. For example, 60° water was analogous to "very warm" water, and 1 cup was analogous to a "small amount." Participants were told to translate the numerical values into descriptions and then estimate the answer by pretending the numbers were not there. After estimating, participants were to compute the answer using math. In the no-elaboration condition, participants were not instructed to make an estimate before computing but were told that thinking about the intuitive task might help them come up with the correct math. Because the instruction conditions did not differ significantly, they are not discussed further.

Scoring of Intuitive Principles

Principle scores were derived from the pattern of final temperature judgments from the intuitive task only. We scored the judgment pattern of the intuitive task for consistency with each of the principles. If a participant understands a principle, then his or her pattern of judgments should be consistent with that principle. In describing the scoring we use the numbers for the temperature and quantity that corresponded to the position of the marker on the ungraduated thermometer and quantity of water. The numbers, although available to the experimenter, were not available to the participants.

Above-Below. Above-Below measures understanding of the principle that the final temperature should always be in the direction of the added temperature from the initial temperature. Six of the added temperature trials were below the initial temperature (20° and 30°), and six were above (50° and 60°). One point was added for each answer on the appropriate side of the initial temperature. One point was subtracted for each answer on the inappropriate side. Nothing was done for ties (i.e., judged final temperature equal to the initial temperature). Forty-degree added temperature trials were not used. The maximum score was 12, and the minimum was −12.

Range. Range measures understanding of the principle that the final temperature must fall between the initial and added temperature, regardless of the quantities. Range was scored by adding one point for every answer between the added and initial temperature. One point was subtracted for each answer not between the added and initial temperature. Nothing was done for ties (i.e., judged final temperature equal to the initial temperature or added temperature). Forty-degree added temperature trials were not used. The maximum score was 12, and the minimum was −12.

Crossover. Crossover measures the understanding of the interaction between temperature and quantity (i.e., the greater the quantity of the added water, the greater the effect of its temperature). Crossover was scored by comparing the ordering of the extreme quantities for each added temperature. For example, the final temperature should be judged colder when 3 cups of 20° are added to the initial temperature than when 1 cup of 20° is added. One point was added for each correct ordering. One point was subtracted for each incorrect ordering. Nothing was done for ties (i.e., final temperature judged to be equal for the compared trials). Forty-degree added temperature trials were not used. The maximum score was 4, and the minimum was −4.

Equal-Temperatures-Equal. ETE measures understanding of the principle that the temperature does not change if added water is the same temperature as the initial temperature. ETE was scored using the 40° added temperature trials. One point was added for each judged final temperature within 1° of 40°. The maximum score was 3, and the minimum was 0.[1]

For ease of comparison, all principle scores were linearly transformed to percentage of the maximum score for that principle.

Scoring of Verbal Protocols

The verbal protocols from the numerical condition were scored for reference to principles and type of strategy used to solve the problem. The scoring was done by James A. Dixon. Two research assistants who were unaware of the predictions of the study coded the protocols from 25 randomly chosen participants. Reliability was .90 as assessed using Cohen's kappa (Siegel & Castellan, 1988).

Results

Replicating previous work (Ahl et al., 1992, Moore et al., 1991; Strauss & Stavy, 1982), we found that performance on the intuitive task was very different from performance on the numerical task. In the intuitive condition even 2nd graders quickly estimated the answer. In the numerical condition, participants took a considerable amount of time to choose and execute a strategy. In the intuitive condition none of the participants appeared to attempt math, and none of the participants spontaneously mentioned using math. In the numerical condition, 78% of the participants attempted some math. The remaining 22% estimated on all trials. These participants did not use explicit calculations despite repeated urging from the experimenter.

The response patterns for each condition also differed dramatically. For example, the average absolute error was 4.22° for the intuitive condition and 16.02° for the numerical condition, $F(1, 115) = 23.13, p < .01$. On average, participants performed much better in the intuitive condition. This result reflects the tendency for estimates to be fairly close to the correct answers even when they systematically violated some of the principles. In contrast, the results of incorrect math strategies often produce answers that are not even close to the correct answer. Brunswik (1956) made a very similar observation.

Reference to Principles

We found that participants do talk about the principles we proposed. These principles were mentioned spontaneously

[1] Although the ETE principle makes a point prediction, we scored responses within 1° of the correct answer as being consistent with the principle. Scoring responses with a more liberal criterion (counting responses within 4° on either side of 40 as correct) does not change any of the results reported here. In fact, the measure we report and the more liberal measure correlate quite highly, $r = .92$.

Table 1
Means and Standard Deviations of Intuitive Principle Scores

Principle	Mention *M*	Mention *SD*	No mention *M*	No mention *SD*
Above–Below	98	4.83	95	9.00
Range	93	11.37	81	23.79
Crossover	86	20.17	81	23.46
Equal–Temperatures–Equal	87	27.42	48	44.79

while participants were attempting to generate math solutions in the numerical condition. The percentage of participants who mentioned each principle was 53%, 45%, 12%, and 5% for ETE, Crossover, Above–Below, and Range, respectively.

Participants who reference a principle while performing the numerical task should show good understanding of that principle as measured by their intuitive principle score. Therefore, for those participants who reference a principle, the mean principle score should be high, and the standard deviation should be low. However, participants who do not reference a principle may or may not understand the principle. Therefore, for those participants who do not reference a principle, the intuitive principle score should be somewhat lower but have high variability.

The means and standard deviations are shown in Table 1. For all principles, the mean intuitive principle score of the Mention group was higher than the mean of the No Mention group, although the difference was significant only for the ETE principle; for Above–Below, $F(1, 28) = 2.62$; for Range, $F(1, 8) = 5.23$; for Crossover, $F(1, 114) = 2.21$; and for ETE, $F(1, 88) = 31.18$ using Welch's correction for unequal variances (Marascuilo & Serlin, 1988).[2] Because the size of the mean difference depends on the likelihood that participants will spontaneously mention a principle, significant differences were not predicted.

Examination of Table 1 shows that the standard deviations were smaller for the Mention groups than for the No Mention groups, as predicted. The variances were significantly different for all principles mentioned except Crossover: $F(101, 13) = 3.47$ for Above–Below; $F(109, 5) = 4.37$ for Range; $F(63, 51) = 1.35$ for Crossover; and $F(54, 60) = 2.67$ for ETE. Participants who mentioned a principle showed good understanding of the principle in the intuitive condition. Relative to the No Mention group, the principle scores of the Mention group have low variability, indicating that they all had similar understanding.

Development of Intuitive Principles

Figure 1 shows the mean principle scores for each age group. The effect of grade is significant for all principles, for Above–Below, $F(4, 111) = 9.32$; for Range, $F(4, 111) = 17.52$; for Crossover, $F(4, 111) = 3.14$; for ETE, $F(4, 111) = 19.57$. Examination of Figure 1 shows a progression in understanding the principles of the domain. Whereas 2nd graders show the poorest understanding of the task, 5th and 8th graders appear to have very similar understanding, as do 11th-grade and college-age participants. Consistent with this interpretation, there are no significant differences between the principles scores for 5th versus 8th graders or for 11th graders versus college students (all *p*s $> .10$). However, there are significant differences when 5th and 8th graders are compared to 11th grade and college participants for Above–Below, $F(1, 103) = 14.92$; for Range, $F(1, 103) = 50.78$; for Crossover, $F(1, 103) = 7.80$; for ETE, $F(1, 103) = 54.38$.

Developmental Differences in the Use of Mathematical Strategies

The vast majority of participants (82%) attempted at least one math strategy in the numerical condition. There was little developmental change in whether participants attempted math. At least one math strategy was used by 91% of second graders, 85% of 5th graders, and approximately 80% of 8th graders, 11th graders, and college students. Participants who did not attempt math in the numerical condition estimated their answers on all trials.

Ten mathematical strategies were identified from the verbal protocols and worksheets. The strategies varied from adding the water temperatures, to the correct strategy of taking the weighted average of the temperatures with the quantities as the weights. The strategies are listed in the body of Table 2. Participants used an average of 1.22 different math strategies across the nine trials ($SD = 0.86$). The number of math strategies used was not significantly related to grade, $F(4, 111) = 2.11, p < .09$. The mean number of math strategies ranged from 1.58 for 5th graders to 0.93 for college students. (Because estimation is not considered a math strategy here, the mean number of math strategies can be less than one.)

It should be noted that in the numerical condition all participants had estimation available to them as a backup strategy if they did not feel comfortable with the math they were generating. A large percentage of participants in each age group estimated on at least one trial (55%, 69%, 65%, 100%, and 93% for 2nd grade through college, respectively). This suggests that participants were willing to use estimation when they were either having trouble generating math or were not satisfied with the math they were using.

We classified each math strategy as being one of four types. Table 2 shows the mathematical strategies and their classifications. Strategies that violate the Range principle (the upper two rows in Table 2) produce answers that are not between the added and initial temperatures. For example, addition of the initial and added temperature always gives an answer that is not between the two temperatures (i.e., violates Range). These strategies also violate Above–Below on half of the relevant trials. Strategies that do not violate Range (the lower two rows in Table 2) produce answers between the initial and added temperatures on all trials and never violate Above–Below. Strategies that ignore quantity (rows 1 and 3 in Table 2) simply do not use the quantities in the calculations. Those that use quantity inappropriately (row 2 in Table 2) use it either as an additive or strictly multiplicative variable. Strategies that use quantity appropriately use it in proportional schemes.

The 3 (Added Temperature) × 3 (Added Quantity) factorial design in the numerical condition produces three types of problems. Although all the problems can be correctly solved with

[2] All significant tests reported are at the $p < .05$ level unless otherwise indicated.

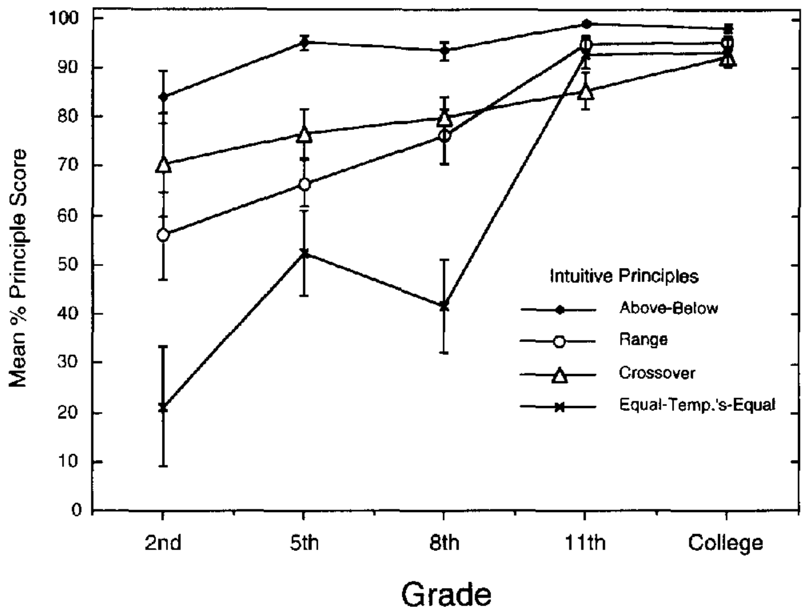

Figure 1. Mean percentage intuitive principle scores plotted as a function of grade with a separate curve for each principle. Bars are ±1 standard error. Temp.'s = temperatures.

weighted averaging, weighted averaging is not always the most efficient strategy. Specifically, when the added and initial water temperatures are equal (Equal Temperature problems) the most efficient procedure is to simply state that the water temperature does not change (although this would not count as a math strategy). Indeed, despite urging from the experimenter, a number of participants would not do math on these types of trials and simply stated the answer without computation. Similarly, when the temperatures are different but the quantities are equal (Equal Quantity–Unequal Temperature problems), the most efficient strategy is to average the temperatures and ignore the quantities. Only when both the temperatures and quantities are unequal (Unequal Quantity–Unequal Temperature problems) is weighted averaging necessary.

Table 3 shows the mean percentage of use for each strategy type by grade and problem type. For each participant we computed the percentage of use for each strategy type by taking the number of times a participant used strategies of that type and dividing it by the total number of trials for that problem type. The percentages in the table are averaged across participants for each grade. To analyze the developmental changes in Table 3, we examined each participant's distribution of strategies across the different trial types. We were interested in two main developmental questions.

1. Do older participants tend to use math strategies that are consistent with range more than younger participants? To address this question we counted the number of trials on which the participant used a math strategy consistent with range and subtracted the number of trials where other math strategies were used. A one-way analysis of variance (ANOVA) on these scores showed a significant effect for grade, $F(4, 111) = 24.21$. As can be seen in Table 3, older participants tend to use math strategies that are consistent with range more often than younger participants. Post hoc analyses show significant differences between the 8th and 11th graders, $F(1, 47) = 12.01$, and 11th graders and college-age students, $F(1, 57) = 5.57$.

2. Do older participants tend to use math strategies which have the appropriate effect of quantity more than younger participants? To address this question, we counted the number of trials on which a math strategy consistent with quantity was used and subtracted the number of trials where other math strategies were used. The effect of grade was significant, $F(4, 111) = 16.11$. Older participants use math strategies that have the appropriate effect of quantity more often than younger participants. Post hoc analyses show significant differences between 8th and 11th graders, $F(1, 47) = 8.92$, and 11th graders and college students, $F(1, 57) = 8.40$.[3]

In summary, there is a developmental progression across grades towards using strategies that do not violate range and that have the appropriate effect of quantity.[4]

Relation Between Intuitive Understanding and Mathematical Strategy Groups

Next, we consider whether the development of intuitive understanding of the problem is related to developmental differ-

[3] In these analyses, we considered the use of unclassifiable strategies as instances of estimation because for most of these trials we were unsure of how the participant arrived at an answer. Most of the trials that were unclassifiable involved a participant unsuccessfully attempting some math and then producing an answer that appeared unrelated to the math. It seems likely that this reflects estimation.

[4] The math strategies we observed for the elementary and high-school students were very consistent with the math strategies which their teachers reported had been taught. Second graders had been taught addition and subtraction. Fifth graders had additionally received some instruction on averaging. Eighth graders had been instructed more thoroughly on averaging. Eleventh graders had received some instruction on weighted averaging.

Table 2
Types of Mathematical Strategies

Strategy type	Strategy
Ignore Quantity–Range Violated	– Add initial and added temperatures. – Subtract initial and added temperatures.
Inappropriate Quantity–Range Violated	– Add initial and added temperatures and quantities. – Add temperature once for each quantity unit. – Multiply quantity by temperature (for added, initial, or both).
Ignore Quantity–Range Not Violated	– Unweighted averaging.
Appropriate Quantity–Range Not Violated	– Weighted averaging. – Take difference between higher and lower temperature. Add proportion of that distance on to lower temperature. Proportion determined by quantity.

ences in generating mathematical strategies. If intuitive understanding is related to generating mathematical strategies, then participants using different mathematical strategies should have different intuitive understanding, as reflected in the participants' principle scores from the intuitive condition.

To test this hypothesis we grouped participants according to the math strategy they used on Unequal Quantity–Unequal Temperature trials. These trials require consideration of both quantity and temperature as explained above. Seventy-three of the 116 participants (63%) used one or more of the math strategies listed in Table 2 on Unequal Quantity–Unequal Temperature trials. Of the remaining participants, 18% estimated on all numerical trials ($N = 21$). About 9% estimated only on Unequal Quantity trials ($N = 10$). An additional 12 participants (10%) used a math strategy that was unclassifiable.

Interestingly, most (82%) of the 73 participants who used one or more math strategies from Table 2 used strategies that came from a single classification. That is, although a participant may have used more than one strategy, all the strategies he or she used were from one of the four types described above (e.g., Ignore Quantity–Violate Range). The remaining 13 participants (18%) who used more than one type of strategy were considered part of the lesser strategy group.[5]

Predicted principle score patterns. If participants use intuitive understanding to guide the generation of mathematical strategies, then participants who use math strategies that violate a principle should show poorer understanding of that principle than participants who use math strategies consistent with that principle. Therefore, participants whose math strategies violate Range (those in the Ignore Quantity–Violate Range and Inappropriate Quantity–Violate Range groups) should show poorer understanding of Range, Above–Below, and ETE than participants whose math strategies do not violate Range (those in the Ignore Quantity–Not Violate Range and Appropriate Quantity–Not Violate Range groups). Strategies that violate Range also violate Above–Below on half the relevant trials and ETE on all the relevant trials. Participants whose math strategies violate Crossover (those in the Ignore Quantity–Violate Range, Inappropriate Quantity–Violate Range, and Ignore Quantity–Not Violate Range groups) should show poorer understanding of Crossover than participants whose math strategies are consistent with Crossover (those in the Appropriate Quantity–Not Violate Range group).

Intuitive principle scores by strategy group. Figure 2 shows the mean intuitive principle scores for each mathematical strategy group. As predicted, the two groups whose math strategies violate Range (Ignore Quantity–Violate Range, $n = 25$, and Inappropriate Quantity–Violate Range, $n = 12$) have lower principle scores for Above–Below, Range, and ETE than the two groups whose math strategies do not violate Range (Ignore Quantity–Not Violate Range, $n = 12$, and Appropriate Quantity–Not Violate Range, $n = 24$): for Above–Below, $F(1, 71) = 14.80$; for Range, $F(1, 71) = 52.83$; and for ETE, $F(1, 71) = 55.77$. Also as predicted, the three groups whose math strategies violate Crossover (Ignore Quantity–Violate Range, Inappropriate Quantity–Violate Range, Ignore Quantity–Not Violate Range) have a lower principle score for Crossover than the group whose math strategies are consistent with Crossover (Appropriate Quantity–Not Violate Range), $F(1, 71) = 13.95$.

Thirty-seven percent of participants did not fit into the mathematical strategy groups presented above. However, the performance of these participants on the numerical task is also consistent with their intuitive understanding. They form three groups, which are presented in Figure 3. First, participants using an Unclassifiable mathematical strategy ($n = 12$) had mean principle scores suggesting good understanding of the domain. However, because their math strategies are unknown, little can be concluded about the relation between intuitive understanding and mathematical strategies. Second, Estimate All Trials participants ($n = 21$) estimated on all the trials and also had good understanding of the domain. The relatively good intuitive understanding of these participants is consistent with the possibility that they were using estimation as a backup strategy because they could not generate appropriate math strategies. The third group, Estimate If Unequal Quantities ($n = 10$), esti-

[5] We grouped these 13 participants by the lesser of their two strategies because of the possibility that understanding of the task might improve while trying to generate a math strategy. It seemed possible that in trying to come up with a math strategy participants might, perhaps through analogy with other tasks, reach a better understanding of the principles. Grouping participants by the better of the two strategies or by the type of strategy they use more often does not change the results substantively.

Table 3
Mean Percentage of Math Strategy Group by Problem Type for Each Grade

	Grade														
	2nd			5th			8th			11th			College		
Strategy	UQ/UT	EQ/UT	ET	UQ/UT	EQ/UT	ET	UQ/UT	EQ/UT	ET	UQ/UT	EQ/UT	ET	UQ/UT	EQ/UT	ET
Ignore Quantity–Range Violated	41	45	39	15	29	10	9	16	3	5	0	1	1	0	0
Inappropriate Quantity–Range Violated	18	23	21	22	16	16	23	24	29	2	2	0	0	0	0
Ignore Quantity–Range Not Violated	0	0	0	1	1	1	11	18	3	9	33	6	8	22	0
Appropriate Quantity–Range Not Violated	0	0	0	0	0	0	4	5	2	18	21	0	47	38	13
Estimate–Other	41	32	40	61	54	73	57	37	63	66	44	93	44	40	87

Note. UQ/UT = trials with unequal quantities and unequal temperatures; EQ/UT = trials with equal quantities and unequal temperatures; and ET = trials with equal temperatures, regardless of quantity.

mated on all trials on which the quantities were not equal. These participants were divided into two subgroups (shown in the right two columns of Figure 3) depending on what type of math they used on Equal Quantity trials. On Equal Quantity trials, 7 participants used math that did not violate Range, and 3 used math that did violate Range. The Range principle score was significantly different for these two groups, in spite of their small number, $F(1, 4) = 47.05, p < .01$.

The pattern of the mean intuitive principle scores for the numerical strategy and estimation groups is consistent with the hypothesis that there is a relation between intuitive understanding and generating mathematical strategies. Developmental differences in math strategy use are related to developmental differences in understanding the principles.

Individual analyses. Although the analysis of the subgroups presented above shows a relation between the development of intuitive understanding and generating math strategies, we were also interested in the nature of the relation. Therefore, we examined individual performance by comparing each participant's understanding of each principle with the type of math strategy they used. Figure 4 shows the observed frequency of participants who either understood or did not understand a principle and whether their math strategies violated the principle. Understanding a principle was defined as having at least 75% of the maximum score for that principle. Participants who estimated on all relevant numerical trials are also shown. For each principle we considered math strategies from all problem types where the principle was applicable.[6] As mentioned above, the majority of participants using math used a strategy or strategies from a single classification. The few participants who used strategies from more than one classification were considered to have used the lesser strategy (see Footnote 5).

The upper left-hand panel of Figure 4 shows the results for the Range principle. The vast majority of participants who used a math strategy that was consistent with Range understood Range. Similarly, most participants who estimated on all the trials where Range was relevant also understood Range. Participants who used a math strategy that violated Range showed a very different pattern. Some of these participants understood Range, whereas others did not. The observed distribution is very unlikely under the hypothesis of no association, $\chi^2(2, N = 109) = 39.19, p < .0001$.

A very similar pattern is seen for the Crossover principle in the lower left-hand panel of Figure 4. This distribution of participants is also very unlikely under a hypothesis of no association, $\chi^2(2, N = 104) = 9.79, p < .008$.

The results for the Above–Below principle are difficult to interpret because almost all participants understood the principle. The distribution of participants across cells is not significantly different from what might be expected under a hypothesis of no association, $\chi^2(2, N = 109) = 5.32, p < .07$.

The results of the ETE principle are in the lower right-hand panel of Figure 4. Almost all participants who used math strategies consistent with this principle understood the principle. Similarly the majority of participants who estimated on equal-temperature trials understood ETE, although the proportion is not as large as for the other principles. Most participants who used a strategy that violated ETE did not understand the principle. The observed distribution is very unlikely under a hypothesis of equal proportions, $\chi^2(2, N = 112) = 29.67, p < .0001$.

In summary, we see two interesting results in the relation between understanding the principles and math strategy use. First, some participants who understood the principle used strategies that violated the principle, and others used strategies consistent with the principle. Second, extremely few participants who did *not* understand a principle used a strategy consistent with it (only 3 participants over all principles). It appears that understanding a principle is necessary but not sufficient to generate a math strategy consistent with that principle. Interestingly, for three of the principles, participants who estimated on all rele-

[6] For the Above–Below and Range principles, math strategies that occurred on both Equal Quantity–Unequal Temperature and Unequal Quantity–Unequal Temperature problems were used. For Crossover, math strategies that occurred on Unequal Quantity–Unequal Temperature problems were used. For the ETE principle, math strategies that occurred on Equal Temperature problems were used.

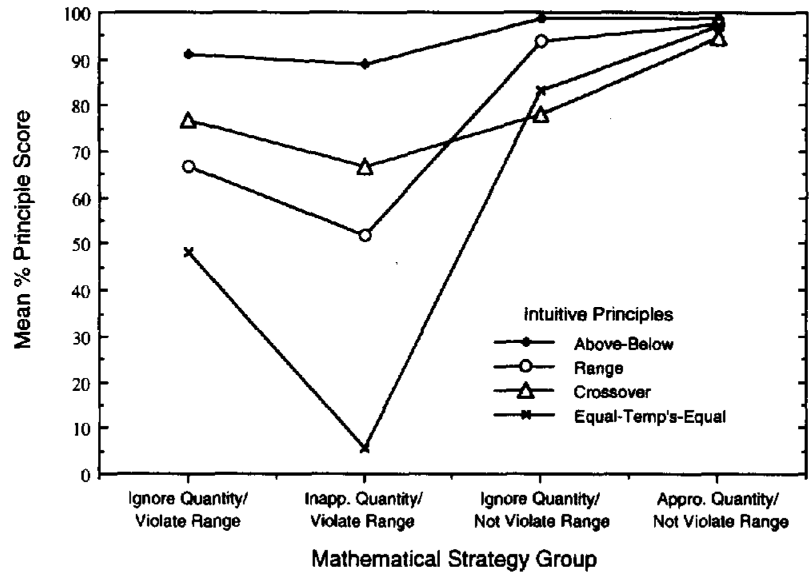

Figure 2. Mean percentage intuitive principle scores plotted as a function of mathematical strategy group with a separate curve for each principle. Inapp. = inappropriate; Appro. = appropriate; Temp's = temperatures.

vant trials showed a similar pattern to participants who used a math strategy consistent with the principle. This suggests that these participants may have been choosing to estimate because they could not generate a math strategy consistent with their understanding of the domain.

Evidence for the Necessity and Insufficiency of Intuitive Understanding Across Developmental Levels

The analyses of Figure 4 aggregated over age groups suggest that understanding a principle is a necessary but not sufficient condition to generate a math strategy consistent with that principle. We are interested in whether we have evidence of both the necessity and insufficiency of understanding a principle for generating an appropriate math strategy at different developmental levels. Evidence for the necessity of understanding a principle would come from a pattern where extremely few participants who did not understand the principle used a math strategy consistent with it. Of course, in order to assess this relation we must observe variability within the age group in both understanding the principles and the type of math strategy used. Evidence for the insufficiency of understanding a principle comes more simply from the presence of participants who understood the principle but did not use a math strategy consistent with it.

For the 2nd graders, we have evidence of the insufficiency of understanding a principle for generating a math strategy consistent with that principle. Across the four principles, an average of 41% of the 2nd graders understood a principle but did not use a math strategy consistent with it.

There is evidence for both the 5th and 8th graders that understanding a principle is necessary to generate an appropriate math strategy but that understanding a principle is not sufficient. For these two age groups very few participants used a math strategy that was consistent with a principle they did not understand, an average of 1% and 3% for 5th and 8th graders, respectively. This result is interpretable because a mean of 6% of 5th graders and 16% of 8th graders used a math strategy that was consistent with a principle they understood. Also, a mean of 26% and 24% of 5th and 8th graders did not understand a principle and used a math strategy that violated that principle.[7]

The 5th and 8th graders also provided evidence that understanding a principle is not sufficient. Across principles, an average of 38% of 5th graders and 25% of 8th graders used a math strategy that violated an understood principle. The 11th graders and college students also provided evidence of the insufficiency of understanding a principle. Across principles, a mean of 12% and 6% of 11th graders and college students, respectively, used a math strategy that violated a principle they understood.

In summary, all age groups contributed evidence that suggests that understanding a principle is not sufficient to generate a math strategy consistent with that principle. However, because evidence for the necessity of understanding a principle requires variability in understanding the principles and in math strategy types, only the 5th and 8th graders provided evidence of necessity.

Modeling the Contribution of Intuitive Understanding and Age on Mathematical Strategy Choice

One interpretation of the observed relation between intuitive understanding and mathematical strategies is that because both

[7] Because fifth and eighth graders had very similar understanding of the principles and showed a very similar relation between understanding a principle and math strategy use, we considered them as a single group to increase statistical power. When fifth and eighth graders are classified by understanding of each principle and by the type of strategy they used (violated the principle, did not violate the principle, estimated all relevant trials), there is a significant relation for Range, $\chi^2(2, N = 42) = 8.89, p < .02$, and ETE, $\chi^2(2, N = 43) = 7.03, p < .03$.

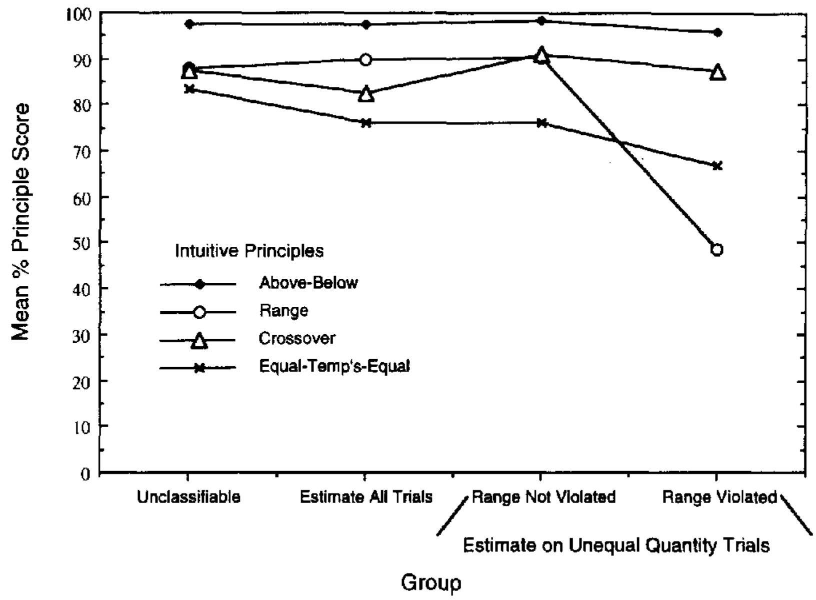

Figure 3. Mean percentage principle scores plotted for unclassifiable and estimation groups with a separate curve for each principle. Participants who estimated on unequal quantity trials (right-hand columns) are shown as two groups based on whether their math strategies on equal quantity trials violated Range. Temp's = temperatures.

are related to age or grade, the observed relation is spurious. According to this argument, intuitive understanding does not directly affect mathematical strategy choice. Rather, they only appear to be linked because both develop with age. As evident in Figure 1 and Table 3, both intuitive principle scores and mathematical strategy choice are correlated with grade. Are intuitive principle scores related to mathematical strategy choice when grade is statistically controlled? To model the contributions of grade and each intuitive principle score in predicting math strategy choice, we performed four stepwise logistic regressions. Logistic regression was used because the dependent variable (i.e., whether or not a math strategy was consistent with the principle) is dichotomous and standard multiple regression requires a continuous dependent variable (Engleman, 1990). The results for logistic regression are interpreted similarly to multiple regression.

In each regression we first entered grade and then entered an intuitive principle score to test for a significant contribution of the principle to predicting whether the participant's math strategy violated the principle or not. Participants whose math was unclassifiable or who estimated on all trials were not included in any regression analyses. For all intuitive principles, the principle score is a significant predictor of math strategy choice after the effect of grade is partialled out: for Above–Below, $b = .14$, $\chi^2(1) = 5.26$; for Range, $b = .06$, $\chi^2(1) = 7.00$; for Crossover, $b = .03$, $\chi^2(1) = 5.02$; and for ETE, $b = .07$, $\chi^2(1) = 6.12$. These results show that, whereas grade is correlated with both the intuitive principle scores and math strategies, there is a relation between the intuitive understanding and math strategy choice that is not accounted for by grade alone.

Discussion

The first goal of the present study was to measure intuitive understanding of the temperature mixture task and to examine whether the proposed principles were used in generating math. Recall that the principle scores were derived from the response pattern in the intuitive condition, but that the references to principles were recorded in the numerical condition while participants were attempting math. The spontaneous verbalization of the principles supports the hypothesis that they are used in choosing math strategies. Some participants described the principles aloud as they attempted to generate the appropriate math. Participants who mentioned a principle tended to have higher scores for that principle in the intuitive task and their scores had significantly less variability. These results suggest that the principle scores may measure the same knowledge that participants spontaneously verbalized. However, it also may be that an underlying representation that does not consist of principles is responsible for both the pattern of judgments in the intuitive condition and the verbalizations in the numerical condition. The question of the nature of the intuitive representation is important if one accepts the possibility that there might be a mapping process taking place between intuitive understanding of the domain and mathematical strategies. The prospect of a mapping process is discussed later.

Understanding of the principles and the sophistication of the math strategies both develop with age. As a group, 2nd graders tend to understand the domain quite poorly. Fifth and 8th graders show better understanding of the principles, although the 8th graders' understanding is not significantly better than that of the 5th graders. The 11th-grade and college-level students understand all the principles at near ceiling levels. There is considerable variability in understanding the principles within age groups, especially for the 2nd, 5th, and 8th graders.

There are also developmental differences in generating mathematical strategies. Second graders use math strategies that violate the Range principle. Similarly, 5th graders most often use math strategies that violate Range. Eighth graders use strategies

	Strategy Consistent with Range 3,4	Strategy Violates Range 1,2	Estimate All Range Trials
Understand Range	46	16	19
Not Understand Range	2	24	2

	Strategy Consistent with Above-Below 3,4	Strategy Violates Above-Below 1,2	Estimate All Above-Below Trials
Understand Above-Below	48	37	21
Not Understand Above-Below	0	3	0

	Strategy Consistent with Crossover 4	Strategy Violates Crossover 1,2,3	Estimate All Crossover Trials
Understand Crossover	24	35	27
Not Understand Crossover	0	14	4

	Strategy Consistent with EQ-T-EQ 3,4	Strategy Violates EQ-T-EQ 1,2	Estimate All EQ-T-EQ Trials
Understand EQ-T-EQ	13	5	47
Not Understand EQ-T-EQ	1	24	22

Figure 4. Frequencies of participants by math strategy and intuitive understanding. The four math strategy types are identified in this figure by the numbers above the columns: (1) Ignore Quantity–Range Violated; (2) Inappropriate Quantity–Range Violated; (3) Ignore Quantity–Range Not Violated; (4) Appropriate Quantity–Range Not Violated. Each panel presents results for a different intuitive principle.

of the 5th graders. The 11th-grade and college-level students understand all the principles at near ceiling levels. There is considerable variability in understanding the principles within age groups, especially for the 2nd, 5th, and 8th graders.

There are also developmental differences in generating mathematical strategies. Second graders use math strategies that violate the Range principle. Similarly, 5th graders most often use math strategies that violate Range. Eighth graders use strategies that violate Range as well but also use strategies consistent with Range and a few use strategies that have the appropriate effect of quantity. The 11th-grade and college students predominantly use strategies that do not violate Range and also use strategies that have the appropriate effect of quantity.

The second goal of the study was to examine whether the developmental differences in intuitive understanding of the domain were related to developmental differences in generating mathematical strategies. The verbal referencing of principles while generating math strategies suggests that there may be a relation between understanding the principles and math strategy generation. Further evidence comes from the pattern of mean intuitive principle scores for the mathematical strategy groups. The mean principle scores showed that participants who used different types of strategies had very different intuitive understandings of the task. The relation was systematic and consistent with the hypothesis that the development of intuitive understanding is an important factor in generating math strategies. By measuring intuitive understanding at a relatively fine grain, as is accomplished by the intuitive principle scores, the systematic relation between the development of intuitive understanding and formal strategies becomes evident. Likewise, participants who systematically estimated on particular trials appeared to be doing so on the basis of what they understood about the task.

Another explanation for the observed relation between intuitive understanding and mathematical strategies is that both intuitive understanding and mathematical strategies are correlated with age and that the relation between intuitive understanding and math strategies is then spurious. Contrary to this explanation, intuitive principle scores are related to math strategy choice when the effect of grade is statistically controlled. This result is consistent with the hypothesis that participants use their intuitive understanding of the task to select math strategies.

Intuitive Understanding Is Necessary But Not Sufficient

How participants use their intuitive understanding of the task to select math strategies is an interesting question. When we compare each participant's understanding of a principle with whether their math strategy was consistent with that principle, we see that understanding the principle seems necessary but not sufficient to arrive at a strategy consistent with that principle. We observed this relation despite the fact that participants always completed the intuitive task before the numerical task and received instructions linking the two tasks.

The conclusion that understanding a principle is necessary but not sufficient to generate a math strategy consistent with that principle has an important developmental implication:

The development of understanding of the problem domain constrains the development of formal problem solving in that domain. If a child must understand a principle of a domain in order to generate a math strategy consistent with that principle, then the development of understanding the principles sets an upper bound on the math strategies the child will spontaneously use. A child will not use a formal solution strategy that represents the problem better than the principles he or she understands. For example, a child who has not yet developed an understanding of the Crossover principle for temperature mixture will not generate weighted averaging for temperature mixture problems. Of course, this does not imply that a child could not be taught to apply any math strategy to a problem. Clearly, a child who did not understand the Crossover principle could be taught to perform weighted averaging for a specified problem (although it would be interesting to see if differences in understanding the principles would contribute to remembering instructed solution strategies). The point here, however, is that intuitive understanding sets an upper bound for generating solutions in novel or uninstructed domains, the situation usually referred to as problem solving.

How Is Intuitive Understanding Used in Math Strategy Generation?

The finding that understanding a principle is a necessary but not sufficient condition for generating a math strategy consistent with that principle suggests that participants perform a mapping process between what they understand about the task and what they understand about different math strategies (Gentner, 1983). That is, participants may take the principles they understand about the task and search for math strategies that instantiate those principles. Our results are consistent with the mapping hypothesis in two ways. First, if participants are performing some sort of mapping between intuitive understanding and math strategies, it seems likely that partial matches, matches based on sharing some attributes but not all, would often be selected. For example, consider a 2nd grader whose understanding of the task can be summarized as "combining things (i.e., both temperatures and amounts) gives you more." Given this understanding of the task, addition of the temperatures provides a partial mapping. A more complete mapping would involve the amounts as well. A major part of the mapping process must be determining which variables require inclusion in the math strategy. Therefore, it is not surprising that many participants use partial matches, and they select math strategies that violate an understood principle.

The second reason our pattern of results fits the mapping hypothesis is that math strategies almost never represent the problem better than the participant's intuitive understanding. For example, consider an 8th-grade participant who does not understand the Range principle. (The mean Range principle score for 8th grade was around 75%, so a good number of them did not understand Range.) One might expect that he or she could arrive at averaging as a math strategy through the association of words in the task (e.g., combine) with past words from math problems (35% of 8th graders did use averaging of some type). However, this does not happen. With only two exceptions, participants who used some sort of averaging scheme (i.e., math strategies that do not violate Range) understood Range. Therefore, at an individual level, the results are consistent with the hypothesis that participants use their intuitive understanding in a mapping process to arrive at a math strategy.

Similar to most work in which complex knowledge structures are presumed to profoundly influence performance (e.g., Chi, Feltovich, & Glaser, 1981; Chi et al., 1982; Cummins et al., 1988; Glenberg & Epstein, 1987; Larkin et al., 1980; Simon & Simon, 1978), we did not attempt to manipulate the participant's knowledge structure. Our present measurement of intuitive knowledge, as opposed to manipulation of it, does not allow us to make conclusions about a causal role in selecting mathematical strategies. However, results from past work support a causal interpretation of intuitive understanding. Ahl et al. (1992) showed that performing the intuitive task before the numerical task improved performance on the numerical task for some age groups. However, performing the numerical task first did not improve performance on the intuitive task. Hardiman et al. (1984) found that giving feedback to participants on their judgments of whether a balance scale would tilt or remain level improved performance on numerically presented weighted averaging problems. Hardiman et al.'s (1984) results show that enhancing intuitive understanding through training improves performance on numerical problems.

The results of our study, taken with the results of Ahl et al. (1992) and Hardiman et al. (1984), offer a first step toward a theory of how intuitive understanding affects the generation of formal representations. The pattern of results suggests that individuals may be performing a mapping between their intuitive understanding and their representation of formal math strategies. The nature of the representations of both the intuitive understanding and math strategies needs to be investigated. Also, the details of how the mapping occurs between the intuitive representation and the formal representations needs to be explored.

The present study shows that developmental differences in understanding the principles of a domain play a central role in explaining developmental differences in math strategies. However, if individuals use a mapping process between intuitive understanding and math strategies to select math strategies, then there are other potential sources of developmental change. There may be developmental differences in the richness of formal strategy (i.e., math) representations. For example, college students may have an elaborate understanding of what multiplication does compared to eighth graders. The sophistication of the formal representation should have a large effect on the goodness of the math solution. If younger individuals have an impoverished representation of how math strategies work, then they will not be able to discriminate appropriate math strategies from inappropriate ones on the basis of a match to intuitive understanding. There also may be developmental change in the mapping process itself. For example, younger individuals may be more likely to select math strategies on the basis of partial matches. Older individuals may be more likely to use the underlying structure of the problem in the mapping process.

An interesting aspect of our results is that individuals sometimes use strategies that violate their intuitive understanding. Presumably, use of a strategy that violates an intuitively understood principle might prompt generation of a new strategy. It would be interesting to examine this process in more detail. Siegler and Jenkins (1989) showed that new strategy construction for addition in very young children may require a large number of trials. The process of new strategy construction for a complex task such as temperature mixture may be accelerated for older individuals because their repertoire of strategies is larger and they may not have to construct a new strategy from scratch.

The present study shows that the development of intuitive understanding and developmental differences in the generation of mathematical strategies are closely related. We present a first step toward a theory of how intuitive understanding affects the generation of formal representations. Future work will show whether the pattern of results observed here holds for strategy generation in other domains. The temperature mixture task, however, is quite similar to word problems used in math texts and encountered in real world situations. Important steps toward a complete theory of the relation between intuitive understanding and the generation of mathematical strategies will include understanding the nature of the intuitive representations and the nature and development of formal strategy representations. When both the intuitive and formal representations are delineated, the development of the mapping process between the two representations can be studied.

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